Solvable Sextic Equations

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Criteria are given for determining whether an irreducible sextic equation with rational coefficients is algebraically solvable over the complex number field $\mathcal{C}$.

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Inroduction

All equations of degree 4 or less have algebraic solutions over $\mathcal{C}$, but from the work of Ruffini, Abel and Galois it is now well-known that all quintic equations do not[1]. One of the first to examine this more closely was D.S. Dummit [2], who in 1991 provided a sixth degree resolvent equation for any quintic and proved that a quintic equation with rational coefficients is solvable if and only if its resolvent has a rational root. He then went on to classify the solvable quintics by giving a procedure for determining their Galois groups. Three years later B.K. Spearman and K.S. Williams [3] used Dummit's results to provide the specific algebraic solution to any solvable quintic. Their proof provides a generalization of Cardano's method for the cubic equation. Specifically, in terms of the Bring-Jerrard form for the general quintic:

> The irreducible polynomial
>
> $$x^5 + ax + b = 0$$
>
> with rational coefficients is solvable by radicals if and only if there exist rational numbers $\epsilon = \pm 1$, $c > 0$ and $e \neq 0$ such that
>
> $$a = \frac{5e^4(3 - 4\epsilon c)}{c^2 + 1} \qquad \text{and} \qquad b = \frac{-4e^5(11\epsilon + 2c)}{c^2 + 1}$$
>
> in which case the roots are
>
> $$x_j = e \sum_{k=1}^{4} \omega^{jk} u_k, \qquad j = 0, 1, 2, 3, 4$$
>
> where $\omega = exp(2\pi i/5)$ and
>
> $$u_1 = (v_1^2 v_3/D^2)^{1/5}, \quad u_2 = (v_3^2 v_4/D^2)^{1/5},$$
>
> $$u_3 = (v_2^2 v_1/D^2)^{1/5}, \quad u_4 = (v_4^2 v_2/D^2)^{1/5}.$$
>
> Here, $D = c^2+1$, $v_1 = \sqrt{D} + \sqrt{D - \epsilon\sqrt{D}}$, $v_2 = -\sqrt{D} - \sqrt{D + \epsilon\sqrt{D}}$, $v_3 = -\sqrt{D} + \sqrt{D + \epsilon\sqrt{D}}$ and $v_4 = \sqrt{D} - \sqrt{D - \epsilon\sqrt{D}}$..

Solvable irreducible quintic equations are rare. For example, with $-40 \leq a, b \leq 40$, apart from the cases where $a = 0$ and b is not a fifth root, there are only six: $x^5 + 20x \pm 32 = 0$, $x^5 + 15x \pm 12 = 0$ and $x^5 - 5x \pm 12 = 0$. The aim of this work is to extend Dummit's analysis to equations of degree six; it is reasonable to expect that solvable irreducible sextics are also rare.

Galois Approach to Degree Six

As is well known, by means of a Tchirnhausen transformation the general sextic

$$x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0 \tag{1}$$

can, in principle, be reduced to the form

$$p(x) = x^6 + x^2 + dx + e = 0 \tag{2}$$

having roots u_j, $j = 1, 2, 3, 4, 5, 6$. Furthermore, any irreducible equation (2) with rational coefficients is solvable by radicals if and only if its Galois group G is a subgroup of one of the two transitive subgroups of S_6: $J = S_2\#S_3$ and $K = S_3\#Z_2$. Here we use $\#$ to denote the wreath product of two groups [4].

Case 1. $G \leq J$**.**

The group J has order 48 and index 15 in S_6 and is generated by the three permutations: (123)(456), (12)(45) and (14). It is straightforward to find the symmetric function

$$\theta_1 = u_1u_2u_3^2u_4u_5u_6^2 + u_1u_2^2u_3u_4u_5^2u_6 + u_1^2u_2u_3u_4^2u_5u_6 \tag{3}$$

which is invariant under J (for which J is the stabilizer). Its conjugates in S_6 are

$$\begin{array}{ll}
\text{in}(45)J & \theta_2 = u_1u_2u_3^2u_4u_5u_6^2 + u_1u_2^2u_3u_4^2u_5u_6 + u_1^2u_2u_3u_4u_5^2u_6 \\
\text{in}(35)J & \theta_3 = u_1u_2u_3^2u_4u_5u_6^2 + u_1^2u_2^2u_3u_4u_5u_6 + u_1u_2u_3u_4^2u_5^2u_6 \\
\text{in}(56)J & \theta_4 = u_1u_2u_3^2u_4u_5^2u_6 + u_1^2u_2u_3u_4^2u_5u_6 + u_1u_2^2u_3u_4u_5u_6^2 \\
\text{in}(46)J & \theta_5 = u_1u_2u_3^2u_4^2u_5u_6 + u_1^2u_2u_3u_4u_5u_6^2 + u_1u_2^2u_3u_4u_5^2u_6 \\
\text{in}(26)J & \theta_6 = u_1u_2^2u_3^2u_4u_5u_6 + u_1^2u_2u_3u_4^2u_5u_6 + u_1u_2u_3u_4u_5^2u_6^2 \\
\text{in}(34)J & \theta_7 = u_1^2u_2u_3^2u_4u_5u_6 + u_1u_2^2u_3u_4u_5^2u_6 + u_1u_2u_3u_4^2u_5u_6^2 \\
\text{in}(16)(24)J & \theta_8 = u_1^2u_2u_3^2u_4u_5u_6 + u_1u_2^2u_3u_4u_5u_6^2 + u_1u_2u_3u_4^2u_5^2u_6 \\
\text{in}(15)(34)J & \theta_9 = u_1u_2u_3^2u_4u_5^2u_6 + u_1^2u_2^2u_3u_4u_5u_6 + u_1u_2u_3u_4^2u_5u_6^2 \\
\text{in}(13)(45)J & \theta_{10} = u_1u_2u_3^2u_4u_5^2u_6 + u_1^2u_2u_3u_4u_5u_6^2 + u_1u_2^2u_3u_4^2u_5u_6 \\
\text{in}(24)(35)J & \theta_{11} = u_1u_2u_3^2u_4^2u_5u_6 + u_1^2u_2^2u_3u_4u_5u_6 + u_1u_2u_3u_4u_5^2u_6^2 \\
\text{in}(23)(45)J & \theta_{12} = u_1u_2u_3^2u_4^2u_5u_6 + u_1^2u_2u_3u_4u_5^2u_6 + u_1u_2^2u_3u_4u_5u_6^2 \\
\text{in}(26)(45)J & \theta_{13} = u_1u_2^2u_3^2u_4u_5u_6 + u_1u_2u_3u_4u_5^2u_6 + u_1u_2u_3u_4^2u_5u_6^2 \\
\text{in}(26)(15)J & \theta_{14} = u_1u_2^2u_3^2u_4u_5u_6 + u_1^2u_2u_3u_4u_5u_6^2 + u_1u_2u_3u_4^2u_5^2u_6 \\
\text{in}(26)(34)J & \theta_{15} = u_1^2u_2u_3^2u_4u_5u_6 + u_1u_2^2u_3u_4^2u_5u_6 + u_1u_2u_3u_4u_5^2u_6^2
\end{array} \tag{4}$$

Since J fixes θ_1 and permutes its conjugates among themselves, the polynomial

$$f(x) = \prod_{i=1}^{15}(x - \theta_i) \tag{5}$$

is J-invariant. If $G \leq J$, then θ_1 is invariant under all the automorphisms of G and is therefore rational. On the other hand, if G is not a subgroup of J, then G contains an automorphism which does not leave θ_1 invariant, so θ_1 will not be rational. Thus, G is J or a subgroup of J if and only if $f(x)$ has at least one root fixed by J, i.e. $f(x)$ has a rational root.

It remains to express the coefficients of $f(x)$ in terms of those of $p(x)$. The former are symmetric functions of the θ_j, which are in turn symmetric functions of the roots u_k. Hence, the coefficients of $f(x)$ are expressible in terms of the elementary symmetric functions of the roots, which are precisely the coefficients of $p(x)$.

As usual, we denote the elementary symmetric function in $y_1, \ldots, y_k$ of degree n by $\sigma_n(y_1, \ldots, y_k))$ and we seek the expression of $\sigma_n(\theta_1, \ldots, \theta_{15})$ in terms of $\sigma_1(u_1, \ldots, u_6) = 0 = \sigma_2(u_1, \ldots, u_6) = \sigma_3(u_1, \ldots, u_6)$, $\sigma_4(u_1, \ldots, u_6) = 1$, $\sigma_5(u_1, \ldots, u_6) = -d$ and $\sigma_6(u_1, \ldots, u_6) = e$. By treating these as functional identities, we simply chose values for the roots to obtain a set of independent linear equations. Since each θ_j is of degree 8, σ_1 is of degree 8, σ_2 of degree 16, etc. Thus, the symmetric functions of the θ's were expressed as linear combinations of terms, of the same degree, of the $\sigma_n(u_1, \ldots, u_6)$. For example,

$$\begin{aligned}\sigma_1(\theta_j) = A_1\sigma_2(u_k)\sigma_6(u_k) + A_2\sigma_3(u_k)\sigma_5(u_k) + A_3\sigma_4^2(u_k) + \\ A_4\sigma_2^2(u_k)\sigma_4(u_k) + A_5\sigma_2(u_k)\sigma_3^2(u_k) + A_6\sigma_2^4(u_k). \end{aligned} \tag{5}$$

This led to sets of linear equations involving as many as 149 unknown A's which we were able to handle using MAPLE. The problem of expressing $\sigma_n(\theta_j)$ in terms of the roots was treated recursively using Newton's formula. This procedure was first carried out for the general sextic (1) which led to an expression for $f(x)$ four pages long, which we shall not reproduce here. For the reduced equation (2) we have

THEOREM 1

The sextic equation $x^6 + x^2 + dx + e = 0$ is solvable by radicals and its Galois group is a subgroup of J if and only if the resolvent equation

$$x^{15} - 6e^2x^{13} - (42e + 3)e^3x^{12} + 7e^4x^{11} + (222e - 21d^2)e^5x^{10} +$$

$$(453e^2 + 57e + 8)e^6x^9 - (340e - 109d^2)e^7x^8 - (1716e^2 - 288d^2e + 17)x^7 -$$

$$(1232e^3 - 300e + 144d^2)e^9x^6 + (1534e^2 + 538d^2e - 353d^4 + 2)e^{10}x^5 +$$

$$(2592e^3 - 96d^2e^2 - 258e + 48d^2)e^{11}x^4 - (1728e^4 + \tag{6}$$
$$1012e^2 - 284d^2e + 94d^4 - 9)e^{12}x^3 + (432e^3 - 2160d^2e^2 + 792d^4e + 118e +$$
$$5d^2)e^{13}x^2 + (1296d^2e^3 - 27e^2 + 138d^2e - 60d^4 - 4)e^{14}x +$$
$$(144d^4e - 32d^6 - 3d^2)e^{15} = 0.$$

The simplest examples can be obtained simply by requiring the constant term on the left hand side of (6) to vanish, which gives

$$e = \frac{32d^4 + 3}{144d^2} \tag{7}$$

Thus, if $d = 1/2$, we find that the Galois group of the irreducible polynomial $36x^6 + 36x^2 + 18x + 5$ is a subgroup of J and is solvable.

Case 2. $G \leq K$

It is easily verified that K is the stabilizer of the symmetric function

$$\phi_1 = u_1u_2u_3^2 + u_1^2u_2u_3 + u_1u_2^2u_3 + u_4u_5u_6^2 + u_4^2u_5u_6 + u_4u_5^2u_6 \tag{8}$$

of the roots. The conjugates of ϕ_1 are

$$\begin{aligned}
\phi_2 &= u_4u_2u_3^2 + u_4^2u_2u_3 + u_4u_2^2u_3 + u_1u_5u_6^2 + u_1^2u_5u_6 + u_1u_5^2u_6 \quad (14)K \\
\phi_3 &= u_5u_2u_3^2 + u_5^2u_2u_3 + u_5u_2^2u_3 + u_4u_1u_6^2 + u_4^2u_1u_6 + u_4u_1^2u_6 \quad (15)K \\
\phi_4 &= u_6u_2u_3^2 + u_6^2u_2u_3 + u_6u_2^2u_3 + u_4u_5u_1^2 + u_4^2u_5u_1 + u_4u_5^2u_1 \quad (16)K \\
\phi_5 &= u_1u_4u_3^2 + u_1^2u_4u_3 + u_1u_4^2u_3 + u_2u_5u_6^2 + u_2^2u_5u_6 + u_2u_5^2u_6 \quad (24)K \\
\phi_6 &= u_1u_5u_3^2 + u_1^2u_5u_3 + u_1u_5^2u_3 + u_4u_2u_6^2 + u_4^2u_2u_6 + u_4u_2^2u_6 \quad (25)K \\
\phi_7 &= u_1u_6u_3^2 + u_1^2u_6u_3 + u_1u_6^2u_3 + u_4u_5u_2^2 + u_4^2u_5u_2 + u_4u_5^2u_2 \quad (26)K \\
\phi_8 &= u_1u_2u_4^2 + u_1^2u_2u_4 + u_1u_2^2u_4 + u_3u_5u_6^2 + u_3^2u_5u_6 + u_3u_5^2u_6 \quad (34)K \\
\phi_9 &= u_1u_2u_5^2 + u_1^2u_2u_5 + u_1u_2^2u_5 + u_4u_3u_6^2 + u_4^2u_3u_6 + u_4u_3^2u_6 \quad (35)K \\
\phi_{10} &= u_1u_2u_6^2 + u_1^2u_2u_6 + u_1u_2^2u_6 + u_4u_5u_3^2 + u_4^2u_5u_3 + u_4u_5^2u_3 \quad (36)K
\end{aligned} \tag{9}$$

As above, we define the polynomial

$$g(x) = \prod_{i=1}^{10}(x - \phi_i). \tag{10}$$

To get the coefficients of $g(x)$ we have followed the procedure described above to express the $\sigma_n(\phi_j)$ in terms of the $\sigma_n(u_k)$. Since the degree of $\sigma_1(u_k)$ is only 4 in this case, the calculations are not as extensive. In this way we have

THEOREM 2

The Galois group of the reduced sextic (2), having rational coefficients, is a subgroup of K if and only if the tenth degree resolvent equation

$$x^{10}+4x^9+6x^8-(66e^2-4)x^7-(324e^2-58d^2e-1)x^6-(642e^2-192d^2e+11d^4)x^5+$$
$$(129e^4-640e^2+246d^2e-22d^4)x^4+(384e^4-74d^2e^3-320e^2+144d^2e-16d^4)x^3+$$
$$(384e^4-108d^2e^3+4d^4e^2-64e^2+32d^2e-4d^4)x^2-(64e^6-128e^4-32d^2e^3+$$
$$40d^4e^2-6d^6e)x-(64e^6-16d^2e^5-64d^2e^3+48d^4e^2-12d^6e+d^8)=0. \tag{10}$$

has a rational root.

Two simple examples are furnished by $d=e=4$ and $d=2=2e$ which give the irreducible solvable equations $x^6-x^2+4x+4=0$ and $x^6-x^2+2x+1=0$ respectively.

Discussion

We can refine the specification of the Galois groups by looking at the discriminant

$$\Delta=\prod_{i\neq j}(u_i-u_j). \tag{11}$$

For the reduced equation (2) this is

$$\Delta=46656e^5+13824e^3-43200d^2e^2+22500d^4e+1024e-3125d^6-256d^2. \tag{12}$$

Thus, G is a subgroup of the alternating group A_6 if and only if $\sqrt{\Delta}$ is rational. Hence if $\sqrt{\Delta}$ is rational the Galois group of $p(x)$ is a subgroup of $L=J\cap A_6$ if and only if $f(x)$ has a rational root and it is a subgroup of $M=K\cap A_6$ if and only if $g(x)$ has a rational root. L has index 30 in S_6 and is generated by (123)(456), (12)(45) and (14)(25). M has index 20 and is generated by (123) and (14)(25)(36). We note also that $G\leq J\cap K$, which is isomorphic to the dihedral group D_6 if both $f(x)$ and $g(x)$ have a rational roots.

We have worked out the resolvent equations $f(x)$ and $g(x)$ for the general sextic (1) (with $a=0$, since this is trivially achieved by a linear substitution) which are available on request. By using different methods, G. W. Smith [5] has identified the Galois groups for several families of sextic equations such as $x^6+(t-6)x^4+(2t-2)x^3+(t+9)x^2+6x+1=0$. We have confirmed his results that $G\leq K$ in this and a few other cases by using the extended form of $g(x)$.

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